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Some remarks on Yang-Baxter algebras*

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Abstract. We point out the existence of an alternative algebraic structure in Yang-Baxter algebra with trigonometric R-matrix, which appears to be the generalization of the Yangian in Yang-Baxter algebras with rational R-matrix and which is described most naturally by q-commutators. Some properties are presented, in particular in the case of the well-known symmetric six-vertex model.

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1 Introduction

Integrable systems in two-dimensional statistical mechanics have statistical weights that satisfy algebraic relations known nowadays as Yang-Baxter equations. These are the necessary conditions which allow to set up a suitable parametrization of the weights leading to the exact evaluation of the partition function of the model in the thermodynamic limit. The Yang-Baxter equations were then recast in an algebraic structure called the quantum inverse scattering method by Faddeev *et al.* in the 70's [1]. The advantage of this approach is the enormous simplification gained in constructing the Bethe-ansatz eigenstates of the row-to-row transfer matrix of the model. However in the middle of the 80's it has been realized that the quantum inverse scattering method does lead to new interesting structures which have helped in understanding various interrelationships among models and their symmetries. Among other topics, the concept of quantum groups which was introduced independently by Jimbo [2] and Drinfeld [3] is in fact already contained in the Yang-Baxter algebra of the quantum inverse scattering method and has helped to understand the degeneracy of the transfer matrix eigenstates, through representation theory of quantum groups [4].

It is remarkable that integrable systems are always linked to algebras of very large dimension: Onsager algebra and Dolan-Grady algebra for the Ising model [5], Virasoros algebras for critical models [6], Temperley-Lieb algebras for vertex-models [7] *etc.* The dimension of these algebras is in fact infinite if one takes the thermodynamic limit. The common feature of all of these models is a very large abelian symmetry generated by all the conserved quantities which are usually obtained from the row-to-row transfer matrix.

Drinfeld has studied in particular the quantization of classical integrable systems (by quantizing the so-called Poisson lie group structures). He realized that the class of solutions found as Yang-Baxter algebras with rational R-matrix leads to a special algebraic structure which he named Yangian [8]. In fact, one may say roughly that the knowledge of the Yangian of a model is equivalent to the knowledge of the Yang-Baxter algebra of the model. A common example of system admitting rational R-matrix is the critical 6-vertex model or equivalently the XXX Heisenberg chain. The R-matrix is a very simple 4×4 matrix with three non-zero matrix elements satisfying a linear relation. The symmetry group is naturally the $sl(2)$ group. Drinfeld has obtained all the algebraic properties of the Yangian and studied some of the representations of Yangians, each representation can be associated to a physical model. Nowadays many Yangians are known for integrable systems such as the Hubbard model, the Bose gas in one dimension *etc.* [9].

The question is now whether a generalization of the concept of Yangian is possible when one has a Yang-Baxter algebra with trigonometric R-matrix. Drinfeld has also raised this issue and has constructed a new algebra $\mathcal{A}$ which he suspected to be linked to Yang-Baxter algebras with trigonometric R-matrix [8]. For about ten years, it has been realized that quantum groups are symmetry groups for the XXZ Hamiltonian with appropriately modified boundary conditions [10]. In fact quantum groups are associated with modified Yang-Baxter algebras with reflection matrices, although a quantum group structure can also be extracted from the Yang-Baxter algebra without modified boundary conditions [1,11].

It is the purpose of this note to explore this problem. We shall start with a concrete Yang-Baxter algebra and reformulate its commutation rules in terms of *q-commutators* which are introduced in the deformation theory of quantum mechanics [12] and group theory to give a more appropriate structure for our purpose. This is

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done in the next section. In Section 3, we shall study the realization of the Yang-Baxter algebra by vertex systems in statistical mechanics starting first with the one-site case which leads to a direct connection with a quantum group. The case of multisite Yang-Baxter algebra is treated next where analyticity with respect to the "spectral parameter" is fully used. The Yangian limit is examined in Section 4. The relation to the dynamics represented by the row-to-row transfer matrix is studied in Section 5. Finally Section 6 is devoted to representations of the new algebra $\mathcal{A}$ with a cyclic vector and in particular in the case of the six-vertex model.

2 Yang-Baxter algebras

We shall consider the set of operators $T(u)$ parametrized by a spectral parameter u fulfilling the relations:

$$R\left(\frac{u}{v}\right) T(u) \otimes T(v) = T(v) \otimes T(u) R\left(\frac{u}{v}\right), \tag{1}$$

where $R\left(\frac{u}{v}\right)$ is a 6-vertex type of R-matrix of the form:

$$R\left(\frac{u}{v}\right) = \begin{pmatrix} \left(\frac{u}{v}q - \frac{v}{u}q^{-1}\right) & 0 & 0 & 0 \\ 0 & (q-q^{-1})\left(\frac{u}{v} - \frac{v}{u}\right) & 0 & 0 \\ 0 & \left(\frac{u}{v} - \frac{v}{u}\right) & (q-q^{-1}) & 0 \\ 0 & 0 & 0 & \left(\frac{u}{v}q - \frac{v}{u}q^{-1}\right) \end{pmatrix} \tag{2}$$

and $T(u)$ is an operator valued 2×2 matrix:

$$T(u) = \begin{pmatrix} A(u) & C(u) \\ B(u) & D(u) \end{pmatrix}. \tag{3}$$

Expanding out the operator equation (1), one obtains 16 relations which are well-known in the formalism of quantum inverse scattering method. However through linear combinations one can rewrite these relations as relations between *q-commutators* for the matrix elements of the monodromy matrix $T(u)$ namely:

$$[A(u), A(v)]=[B(u), B(v)]=[C(u), C(v)]=[D(u), A(v)]=0. \tag{4}$$

Moreover if we define the *q-commutator* of two operators M and N as:

$$[M, N]_q = (MN - qNM). \tag{5a}$$

Note that:

$$[M, N]_{q=1} = [M, N] \quad \text{and} \quad [M, N]_{q=-1} = \{M, N\}. \tag{5b}$$

Then we have 8 relations for *q-commutators*:

$$\begin{aligned}
[A(u), B(v)]_q &= \left(\frac{v}{u}\right) [A(v), B(u)]_q, \\
[B(v), A(u)]_q &= \left(\frac{u}{v}\right) [B(u), A(v)]_q, \\
[A(u), C(v)]_q &= \left(\frac{u}{v}\right) [A(v), C(u)]_q, \\
[C(v), A(u)]_q &= \left(\frac{v}{u}\right) [C(u), A(v)]_q, \\
[D(u), B(v)]_q &= \left(\frac{u}{v}\right) [D(v), B(u)]_q, \\
[B(v), D(u)]_q &= \left(\frac{v}{u}\right) [B(u), D(v)]_q, \\
[D(u), C(v)]_q &= \left(\frac{v}{u}\right) [D(v), C(u)]_q, \\
[C(v), D(u)]_q &= \left(\frac{u}{v}\right) [C(u), D(v)]_q.
\end{aligned} \tag{6}$$

Among diagonal and antidiagonal elements of $T(u)$ we have:

$$\begin{aligned}
[A(u), D(v)] &= [A(v), D(u)], \\
[B(u), C(v)] &= [B(v), C(u)].
\end{aligned} \tag{7}$$

Also there are two additional relations:

$$\begin{aligned}
&(q-q^{-1})(\{A(u), D(v)\} - \{A(v), D(u)\}) = \\
&\qquad 2\left(\frac{v}{u} - \frac{u}{v}\right)([B(u), C(v)]), \\
&(q-q^{-1})(\{B(u), C(v)\} - \{B(v), C(u)\}) = \\
&\qquad 2\left(\frac{v}{u} - \frac{u}{v}\right)([A(u), D(v)]).
\end{aligned} \tag{8}$$

This formulation of the Yang-Baxter algebra in terms of q-commutators seems natural if one seeks to reveal the hidden quantum group structure.

3 Realizations by lattice vertex systems

3.1 The one site Yang-Baxter algebra

This is the case of one vertex operator whereby one takes, following [13]:

$$T(u) = \begin{pmatrix} A(u) = (uq^H - u^{-1}q^{-H}) & C(u) = (q-q^{-1})J^- \\ B(u) = (q-q^{-1})J^+ & D(u) = (uq^{-H} - u^{-1}q^H) \end{pmatrix}, \tag{9}$$

where the operators H, J^+, J^- are the generators. The set of previous commutators (4, 6, 7, 8) reduces then to the three simple ones:

$$\begin{aligned}
[q^H, J^+]_q &= (q^H J^+ - qJ^+ q^H) = 0 \\
[q^{-H}, J^-]_q &= (q^{-H} J^- - qJ^- q^{-H}) = 0 \\
[J^+, J^-] &= \frac{q^{2H} - q^{-2H}}{q - q^{-1}}.
\end{aligned} \tag{10}$$

$$(q-q^{-1})(\{A_{2M+1}(q),D_{-2M-1}(q)\}-\{A_{-2M-1}(q),D_{2M+1}(q)\})=-2[B_{2M}(q),C_{-2M}(q)]$$
$$(q-q^{-1})(\{A_{-2M-1}(q),D_{2M+1}(q)\}-\{A_{2M+1}(q),D_{-2M-1}(q)\})=+2[B_{-2M}(q),C_{2M}(q)]. \quad (14b)$$

These are precisely those of the $U_q(sl(2))$ quantum group. As in [13], we notice that for one site $A(u)$ and $D(u)$ are linear in u and u^{-1} whereas $B(u)$ and $C(u)$ are just constant operators. When $q \to 1$ we recover the $sl(2)$ algebra.

3.2 Multisite Yang-Baxter algebra

When N vertices are disposed on the line we can generate a monodromy matrix $T(u)$, the elements of which have the following u-expansion:

$$T(u)=\begin{pmatrix}A(u) & C(u)\\ B(u) & D(u)\end{pmatrix}= \quad (11)$$
$$\begin{pmatrix}\sum_{m=-M-1}^{M}u^{2m+1}A_{2m+1}(q) & \sum_{m=-M}^{M}u^{2m}C_{2m}(q)\\ \sum_{m=-M}^{M}u^{2m}B_{2m}(q) & \sum_{m=-M-1}^{M}u^{2m+1}D_{2M+1}(q)\end{pmatrix}.$$

This form of $T(u)$ can be most easily recognized by calculating explicitly $T(u)$ for $M=1,2$ using the one vertex representation given by equation (9).

Here we have assumed for clarity $N=2M+1$, so that the total number of operators is $2(2N+1)$; they obey the following set of relations among commutators:

$$[A_{2m+1}(q),A_{2n+1}(q)]=[B_{2m}(q),B_{2n}(q)]=$$
$$[C_{2m}(q),C_{2n}(q)]=[D_{2m+1}(q),D_{2n+1}(q)]=0,$$

and *q-commutators*:

$$[A_{2m+1}(q),B_{2n}(q)]_q=[A_{2n+1}(q),B_{2m+2}(q)]_q,$$
$$[B_{2n}(q),A_{2m+1}(q)]_q=[B_{2m}(q),A_{2n+1}(q)]_q,$$
$$[A_{2m+1}(q),C_{2n}(q)]_q=[A_{2n+1}(q),C_{2m}(q)]_q,$$
$$[C_{2n}(q),A_{2m+1}(q)]_q=[C_{2m+2}(q),A_{2n-1}(q)]_q,$$
$$[D_{2m+1}(q),B_{2n}(q)]_q=[D_{2n+1}(q),B_{2m}(q)]_q,$$
$$[B_{2n}(q),D_{2m+1}(q)]_q=[B_{2m+2}(q),D_{2n-1}(q)]_q,$$
$$[D_{2m+1}(q),C_{2n}(q)]_q=[D_{2n-1}(q),C_{2m+2}(q)]_q,$$
$$[C_{2n}(q),D_{2m+1}(q)]_q=[C_{2m}(q),D_{2n+1}(q)]_q. \quad (12)$$

The last 4 commutators containing the pairs $(A(u),D(v))$ and $(B(u),C(v))$ are equivalent to:

$$[A_{2m+1}(q),D_{2n+1}(q)]=[A_{2n+1}(q),D_{2m+1}(q)],$$
$$[B_{2m}(q),C_{2n}(q)]=[B_{2n}(q),C_{2m}(q)]. \quad (13a)$$

$$(q-q^{-1})(\{A_{2m+1}(q),D_{2n+1}(q)\}-\{A_{2n+1}(q),D_{2m+1}(q)\})$$
$$=2[B_{2m+2}(q),C_{2n}(q)]-2[B_{2m}(q),C_{2n+2}(q)],$$

$$(q-q^{-1})(\{B_{2m}(q),C_{2n}(q)\}-\{B_{2n}(q),C_{2m}(q)\})$$
$$=2[A_{2m+1}(q),D_{2n-1}(q)]-2[A_{2m-1}(q),D_{2n+1}(q)]. \quad (13b)$$

These relations take particular forms whenever the indices reach the ends of their ranges since some of the operators are equal to 0:

$$A_{2M+3}(q)=A_{-2M-3}(q)=B_{2M+2}(q)=B_{-2M-2}(q)=0,$$
$$C_{2M+2}(q)=C_{-2M-2}(q)=D_{2M+3}(q)=D_{-2M-3}(q)=0,$$

as well as for higher indices. It results that one has for $n=-M$, ..., M:

$$[A_{2M+1}(q),B_{2n}(q)]_q=[B_{2n}(q),A_{-2M-1}(q)]_q=0,$$
$$[C_{2n}(q),A_{2M+1}(q)]_q=[A_{-2M-1}(q),C_{2n}(q)]_q=0,$$
$$[D_{-2M-1}(q),B_{2n}(q)]_q=[B_{2n}(q),D_{2M+1}(q)]_q=0,$$
$$[C_{2n}(q),D_{-2M-1}(q)]_q=[D_{2M+1}(q),C_{2n}(q)]_q=0. \quad (14a)$$

It remains to examine the limiting cases of equations (8). Inspection shows that

See equations (14b) above.

Now by construction we have:

$$A_{\pm(2M+1)}(q)=(-1)^{(2M+1)}q^{\pm J^z},$$
$$A_{\pm(2M+1)}(q)D_{\pm(2M+1)}(q)=I$$
$$\text{and}\quad J^z=\sum_{j=1}^{2M+1}H_j.$$

We may relabel the other relevant operators as:

$$B_{\pm 2M}(q)=(q-q^{-1})J^+(q^\pm)$$
$$\text{and}\quad C_{\pm 2M}(q)=(q-q^{-1})J^-(q^\pm).$$

Equations (14b) together with (14a) for $n=-M$, M yield the commutation relations of the quantum group $(sl(2))$:

$$[J^+(q^\pm),J^-(q^\pm)]=\frac{q^{2J^z}-q^{-2J^z})}{(q-q^{-1})},$$
$$q^{J^z}J^+(q^\pm)=qJ^+(q^\pm)q^{J^z},$$
$$J^-(q^\pm)q^{J^z}=qq^{J^z}J^-(q^\pm).$$

Thus this algebra $\mathcal{A}$ is a larger structure which contains the quantum group as a subset, its commutation rules extend naturally those of the quantum group as fixed by the Yang-Baxter algebra.

4 The Yangian limit of the algebra $\mathcal{A}$

As suggested by Sklyanin [18], one may say that the R-matrix of equation (2) is the structure contants tensor of the algebra $\mathcal{A}$. If R depends upon a parameter ε, then the limit $\varepsilon \to 0$ defines a new R-matrix $R(0)$ hence a new algebra which may thought as contraction of the original algebra. Hereafter we show that the limit of $\mathcal{A}$ as $q \to 1$ is the Yangian algebra $\mathcal{Y}(sl(2))$ of Drinfeld under the form studied by Kirillov and Reshetikhin [14].

Setting $q = \exp \varepsilon\lambda$, $u = \exp \varepsilon\theta$ and $v = \exp \varepsilon\theta'$ we find that

$$\frac{1}{2\varepsilon} R\left(\frac{u}{v}\right) \underset{\varepsilon\to 0}{\longrightarrow} R(\theta-\theta') = \begin{pmatrix} (\theta-\theta'+\lambda) & 0 & 0 & 0 \\ 0 & \lambda & (\theta-\theta') & 0 \\ 0 & (\theta-\theta') & \lambda & 0 \\ 0 & 0 & 0 & (\theta-\theta'+\lambda) \end{pmatrix}. \quad (15)$$

This R-matrix is called a rational R-matrix; it defines a new algebra for the monodromy matrix $T(\theta)$:

$$T(\theta) = \begin{pmatrix} A(\theta) & C(\theta) \\ B(\theta) & D(\theta) \end{pmatrix}. \quad (16)$$

The set of Yang-Baxter equations reads now

$$R(\theta-\theta')T(\theta) \otimes T(\theta') = T(\theta)R(\theta-\theta'),$$

and is equivalent to the limiting form of equations (4, 6, 7, 8).

Since the commutator and anticommutator structures remain insensitive to the limit of $\varepsilon \to 0$, equations (4,7) do not change form and the variables (u, v) are replaced by (θ, θ'):

$$[A(\theta), A(\theta')] = [B(\theta), B(\theta')] = [C(\theta), C(\theta')] = [D(\theta), D(\theta')] = 0, \quad (4')$$

$$[A(\theta), D(\theta')] = [A(\theta'), D(\theta)]$$
$$[B(\theta), C(\theta')] = [B(\theta'), C(\theta)]. \quad (7')$$

The limiting form of equation (8) is obtained by taking terms of order ε:

$$\lambda(\{A(\theta), D(\theta')\} - \{A(\theta'), D(\theta)\}) = 2(\theta'-\theta)([B(\theta), C(\theta')]),$$
$$\lambda(\{B(\theta), C(\theta')\} - \{B(\theta'), C(\theta)\}) = 2(\theta-\theta')([A(\theta), D(\theta')]). \quad (8')$$

Now observing that

$$\lim_{\varepsilon\to 0} [M, N]_q = [M, N],$$
$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}([M, N]_q - [M, N]) = \lambda NM,$$

we see that equations (6) collapse to 4 equations:

$$[A(\theta), B(\theta')] = [A(\theta'), B(\theta)],$$
$$[A(\theta), C(\theta')] = [A(\theta'), C(\theta)],$$
$$[D(\theta), B(\theta')] = [D(\theta'), B(\theta)],$$
$$[D(\theta), C(\theta')] = [D(\theta'), C(\theta)]. \quad (6')$$

But as in equation (8) we must keep also terms of order ε, they yield 8 equations:

$$\lambda(B(\theta)A(\theta') - B(\theta')A(\theta)) = (\theta'-\theta)[A(\theta), B(\theta)],$$
$$\lambda(A(\theta')B(\theta) - A(\theta)B(\theta')) = (\theta-\theta')[B(\theta), A(\theta')],$$
$$\lambda(C(\theta)A(\theta') - C(\theta')A(\theta)) = (\theta-\theta')[A(\theta'), C(\theta)],$$
$$\lambda(A(\theta')C(\theta) - A(\theta)C(\theta')) = (\theta'-\theta)[C(\theta), A(\theta')],$$
$$\lambda(B(\theta)D(\theta') - B(\theta')D(\theta)) = (\theta-\theta')[D(\theta'), B(\theta)],$$
$$\lambda(D(\theta')B(\theta) - D(\theta)B(\theta')) = (\theta'-\theta)[B(\theta), D(\theta')],$$
$$\lambda(C(\theta)D(\theta') - C(\theta')D(\theta)) = (\theta'-\theta)[D(\theta'), C(\theta)],$$
$$\lambda(D(\theta')C(\theta) - D(\theta)C(\theta')) = (\theta-\theta')[C(\theta), D(\theta')]. \quad (6'')$$

Equations (6″) are formed by 4 pairs, an equation of any pair may be obtained from the other equation of the same pair through relabelling and use of the corresponding equation in (6′).

At this stage we recognize the definition of the Yangian $\mathcal{Y}(sl(2))$ algebra as formulated in reference [14] (see Eq. (2.1) therein); their (u, v) variables are related to our (θ, θ') by $u = (\theta - \theta')$ and $v = \theta'$, with $\lambda = 1$.

4.1 The one-site algebra solution

It is obtained as limit, for $\varepsilon \to 0$, of the Wiegmann-Zabrodin construction (see Eq. (9)):

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} T(u) = \begin{pmatrix} A(\theta) = (\theta+\lambda H) & C(\theta) = \lambda J^- \\ B(\theta) = \lambda J^+ & D = (\theta - \lambda H) \end{pmatrix}, \quad (9')$$

then the set of relations (4′, 6′, 6″, 7′, 8′) reduces to the commutation relations for the $sl(2)$ algebra:

$$[H, J^+] = J^+ \quad [H, J^-] = -J^- \quad [J^+, J^-] = 2H. \quad (10')$$

This is the expected result from the limit $q \to 1$ of equations (10).

4.2 The multisite algebra $\mathcal{Y}$(sl(2))

A multisite construction may be given from the one site construction by considering the $N(= 2M+1)$ fold coproduct of one site monodromy matrices. In equation (11) the u parameter is replaced by the θ parameter as expansion parameter for the elements of the multisite monodromy matrix. We observe however that θ appears now with odd and even powers in the expansions:

$$T(\theta) = \begin{pmatrix} A(\theta) = \sum_{m=0}^{N} \theta^m A_m(\lambda) & C(\theta) = \sum_{m=0}^{N-1} \theta^m C_m(\lambda) \\ B(\theta) = \sum_{m=0}^{N-1} \theta^m B_m(\lambda) & D(\theta) = \sum_{m=0}^{N} \theta^m D_m(\lambda) \end{pmatrix}. \quad (11')$$

In particular we note that here the operators $A_N(\lambda) = D_N(\lambda) = I$ are just trivial. The form given by Kirillov and Reshetikhin corresponds in fact to the scaled monodromy matrix $\theta^{--N}T(\theta)$ [14].

Equations (4′, 6′, 6″, 7′, 8′) yield consequently the commutators for the generators $A_m(\lambda)$, $B_{m'}(\lambda)$, $C_n(\lambda)$, $D_{n'}(\lambda)$ of the algebra $\mathcal{Y}(sl(2))$, which shall be written for simplicity as A_m, $B_{m'}$, C_n, $D_{n'}$ in the remaining part of this section:

$$[A_m, A_n] = [B'_m, B_n] = [C_m, C_n] = [D_m, D_n] = 0, \quad (4'')$$

$$\begin{aligned}
\lambda(B_mA_n - B_nA_m) &= [A_m, B_{n-1}] - [A_{m-1}, B_n],\\
\lambda(A_mB_n - A_nB_m) &= [B_{m-1}, A_m] - [B_n, A_{m-1}],\\
\lambda(C_nA_m - C_mA_n) &= [A_{n-1}, C_m] - [A_n, C_{m-1}],\\
\lambda(A_nC_m - A_mC_n) &= [C_m, A_{n-1}] - [C_{m-1}, A_n],\\
\lambda(B_nD_m - B_mD_n) &= [D_m, B_{n-1}] - [D_{m-1}, B_n],\\
\lambda(D_nB_m - D_mB_n) &= [B_{n-1}, D_m] - [B_n, D_{m-1}],\\
\lambda(C_nD_m - C_mD_n) &= [D_{m-1}, C_n] - [D_m, C_{n-1}],\\
\lambda(D_nC_m - D_mC_n) &= [C_{m-1}, D_n] - [C_m, D_{n-1}]. \quad (6''')
\end{aligned}$$

$$[A_m, D_n] = [A_n, D_m], \quad [B_m, C_n] = [B_n, C_m]. \quad (7'')$$

$$\begin{aligned}
\lambda(\{B_n, C_m\} - \{B_m, C_n\}) &= 2[A_n, D_{m-1}] - 2[A_{n-1}, D_m],\\
\lambda(\{A_n, D_m\} - \{A_m, D_n\}) &= 2[B_n, C_{m-1}] - 2[B_{n-1}, C_m]. \quad (8'')
\end{aligned}$$

Keeping in mind that $A_N = D_N = I, B_N = C_N = 0$ by construction, we observe that the previous commutators (6‴, 8″) for end values of indices m, n yield the commutators of the $sl(2)$ algebra for the operators:

$$(A_{N-1} - D_{N-1}, B_{N-1}, C_{N-1}),$$

namely:

$$\begin{aligned}
[A_{N-1} - D_{N-1}, B_{N-1}] &= 2\lambda B_{N-1},\\
[A_{N-1} - D_{N-1}, C_{N-1}] &= -2\lambda C_{N-1},\\
[B_{N-1}, D_{N-1}] &= \lambda(A_{N-1} - D_{N-1}).
\end{aligned}$$

This is, up to some rescaling, the expected result from the $q \to 1$ limit of $\mathcal{A}$.

5 Relation to the dynamics

Let us define the following linear combinations of $A(u)$ and $D(u)$:

$$\begin{aligned}
\tau_q(u) &= qA(u) + q^{-1}D(u)\\
\tau_{q^{-1}}(u) &= q^{-1}A(u) + qD(u)\\
\tau_{-1}(u) &= A(u) - D(u)\\
\tau_q(u) &= A(u) + D(u). \quad (17)
\end{aligned}$$

The last operator $\tau(u)$ is just the row-to-row transfer matrix of the $N = 2M + 1$ sites vertex model. It defines the "dynamics" or Euclidean time evolution perpendicular to the row of vertices. The first equation (7) shows that $\tau(u)$ form a commuting family with respect to the parameter u.

Here we would like to show that $A_{2M+1}(D_{-2M-1})$, $A_{-2M-1}(D_{2M+1})$, B_{2M}, C_{-2M} generators of $U_q(sl(2))$ do not all commute with $\tau(u)$. Indeed starting from the relations (6) we may obtain:

$$\left(\frac{y}{v} - \frac{v}{u}\right)\tau(u)C(v) = C(v)\left(\frac{u}{v}\tau_q(u) - \frac{v}{u}\tau_{q^{-1}}(u)\right) - (q - q^{-1})C(u)\tau_{-1}(v) \quad (18)$$

and using the previous commutators (12–15) we are led to the relations:

$$\begin{aligned}
u\tau(u)C_{2m}(q) - u^{-1}\tau(u)C_{2m-2}(q) &=\\
C_{2m}(q)u\tau_q(u) - C_{2m-2}u^{-1}\tau_{q^{-1}}(u)&\\
- (q - q^{-1})C(u)\tau_{-1,2m-1}& \quad (19)
\end{aligned}$$

especially for $m = M + 1$ we get:

$$\tau(u)C_{2M}(q) = (C_{2M}(q)\tau_{q^{-1}}(u) + (q - q^{-1})uC(u)\tau_{-1,2M+1}). \quad (20)$$

This shows that $C_{2M}(q)$ does not commute with the transfer matrix $\tau(u)$ except when $q \to 1$, *i.e.* in the case of the rational R-matrix.

However one can show using the same approach that:

$$\begin{aligned}
A_{2M+1}(q)D(u) - D(u)A_{2M+1}(q) &= 0\\
A_{2M+1}(q)A(u) - A(u)A_{2M+1}(q) &= 0.
\end{aligned}$$

A similar result holds for $D_{2M+1}(q)$ indicating that there is a very *reduced* symmetry for the dynamics .

Finally it is known that the $U_q(sl((2))$ is only a symmetry group when appropriate boundary conditions are met and this is the case with reflexion matrices introduced by Sklyanin [11].

6 Spin 1/2 cyclic representation

In this section we examine some features related to the spin 1/2 representation of this algebra $\mathcal{A}$ which correspond to the symmetric 6-vertex model. As it is known in the theory of quantum inverse scattering for integrable systems there exists a cyclic representation generated from a bare vacuum vector $|\Omega\rangle$ made up of the tensor product of up-spins on each lattice site:

$$|\Omega\rangle = |\uparrow_1\uparrow_2 \cdots \uparrow_N\rangle.$$

The representation is defined by the following choice of generators of $sl(2)$:

$$J^- = \sigma^-, J^+ = \sigma^+, H = \left(\frac{1}{2}\right)\sigma^z \quad (22)$$

in terms of Pauli matrices. It is known that in the usual quantum inverse scattering method the state of one down-spin is created by the $C(u)$ operator applied to the vacuum generating a one-particle wave function:

$$\begin{aligned}
&C(u)[\Omega\rangle\\
&= \sum_{j=1}^{N}(u\sqrt{q} - u^{-1}\sqrt{q^{-1}})^{j-1}(q - q^{-1})\\
&\times (u\sqrt{q^{-1}} - u^{-1}\sqrt{q})^{N-j}|\uparrow_1\uparrow_2 \cdots \downarrow_j \cdots \uparrow_N\rangle. \quad (23)
\end{aligned}$$

Thus the one down-spin has a wavefunction:

$$\psi_j(u,q) = (q-q^{-1})\left(u\sqrt{q} - \frac{1}{u\sqrt{q}}\right)^{j-1}\left(\frac{u}{\sqrt{q}} - \frac{\sqrt{q}}{u}\right)^{N-j} \tag{24}$$

which is essentially a plane wave at space position j.

Now in the new alternative formulation the $C_{2m}(q)$ operators when applied to the vacuum $|\Omega\rangle$ will generate a one down-spin state with a polynomial wave function in q:

$$C_{2(M-p)}(q)|\Omega\rangle = \sum_{j=1}^{N} T_j(p,q)|\uparrow_1\uparrow_2\cdots\downarrow_j\cdots\uparrow_N\rangle \tag{25}$$

with wavefunction at site j:

$$T_j(p,q) = (-1)^p q^{-(M+1)+j}(q-q^{-1}) \sum_{r+s=p} \binom{j-1}{r}\binom{N-j}{s} q^{z-r}. \tag{26}$$

Recall that $N = 2M+1$. These polynomials have obviously a generating function:

$$(q-q^{-1})\Big(u\sqrt{q} - \frac{1}{u\sqrt{q}}\Big)^{j-1}\Big(\frac{u}{\sqrt{q}} - \frac{\sqrt{q}}{u}\Big)^{N-j} = \sum_{p=M-1}^{M} u^{2(M-p)} T_j(p,q). \tag{27}$$

Basically the $C_{2m}(q)$ will play the same role as before: flipping down up-spins whereas the $B_{2m}(q)$ flip them up. The total number of down-spins remain conserved by the $A_{2m+1}(q), D_{2m+1}(q)$ operators.

However the situation is a bit more complicated in the sector of two down-spins in quantum inverse scattering the application of two $C(u)$ operators generates automatically a Bethe-ansatz wavefunction for two down-spins:

$$C(u')C(u)|\Omega\rangle = \sum_{l\leq j}\Big\{\frac{(q\frac{u'}{u} - q^{-1}\frac{u}{u'})}{(\frac{u'}{u} - \frac{u}{u'})}\psi_l(u',q)\psi_j(u,q) + \frac{(\frac{u'}{u} - q^{-1}\frac{u'}{u})}{(\frac{u}{u'} - \frac{u'}{u})}\psi_l(u,q)\psi_j(u',q)\Big\}|\uparrow_1\cdots\downarrow_l\cdots\downarrow_j\cdots\uparrow_N\rangle. \tag{28}$$

Now in the other formalism we have:

$$C_{2(M-p)}(q)C_{2(M-p')}(q)|\Omega\rangle = \sum_{l\leq l\leq j\leq N}\Big(T_{j-1}(p,q)T_{l-1}(p',q) + T_j(p',q)T_l(p,q) + T_{l\leq j}(p,p',q))|\uparrow_1\cdots\downarrow_l\cdots\downarrow_j\cdots\uparrow_N\rangle \tag{29}$$

where the two down-spins have in addition of symmetrized product of single down-spin wavefunctions also a true pair wavefunction:

$$T_{l\leq j}(p,p',q)) = [(-1)^p q^{-(M+1)+l}(q-q^{-1})][(-1)^{p'} q^{-(M+1)+j}(q-q^{-1})] \times \sum_{s+t=p,s'+t=p'} q^{t-s}q^{t'-s'} \times \sum_{n=l+1,\cdots,j-1}\binom{n-1}{s}\binom{N-n}{t}\binom{l+j-n-2}{s'}\binom{N-l-j-1+n}{t'}. \tag{30}$$

It is remarkable that the one down-spin has a polynomial wave function as in the case of the generator of the corner transfer matrix [15], moreover the structure of equation (29) is reminiscent of that of the two down-spins in corner transfer matrix found by Davies about a decade ago [16].

7 Conclusion

In this short note we have shown that a new algebraic structure $\mathcal{A}$ exists for Yang-Baxter algebras with trigonometric R-matrix. This structure contains the usual $U_q(sl(2))$ and generalizes the concept of *Yangian* found in Yang-Baxter algebras with rational R-matrix. We note that this new algebra is not a symmetry algebra of the related row-to-row transfer matrix, however exhibits interesting features which seem to be related to those of corner transfer matrices. Several relevant questions may be raised at this point:

- Does an elliptic version of $\mathcal{A}$ exist?
- Can it help in constructing eigenstates of the row-to-row transfer matrix?
- What is the relation to the Corner Transfer Matrix [17]?

We hope to be able to tackle them in the near future.

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